

General representation of collective neural dynamics with columnar modularity

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We exhibit a mathematical framework to represent the neural process at the cortical level. The description of neural dynamics with columnar and functional modularity, named the fibre bundle map (FBM) representation method, is based on both neuroscience and informatics, whereas it leads to the conventional formulas in statistical physics. The possibility of analogy between the phenomena in brain and physical systems has been proposed [1, 2]. In spite of the complex circuitry and the nonlinear dynamics in neural systems, the neural behavior at high levels may be described by simple and general rules, which related to the noble theory in statistical physics. The FBM method gives profit in building or analyzing the neural models by representing essential ingredients of neural interactions by general formulas. We apply our method to the proposed models of visual map formation and show how they can share statistical properties with vortex dynamics in magnetism in spite of various development mechanisms.

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I. INTRODUCTION

Though the detailed dynamics of a single neuron are revealed, there still remains a challenge at the network level to explain how brains perform higher cognitive functions. Studies of physical models are focused on achieving a biological realism of the neural computation models. However, the success of the basic neural network models, based on the connectional framework between simple cells, in the application of small adaptive systems, they are inherently problematic in the apprehension of collective neural phenomena and higher cognitive behavior in the real brain. And also there are attempts to see through the neural processing at higher levels, the functional modularity of neurons or the symbolic processing architecture. Before the physiological evidence of repetitive cortical blocks, there were proposals of the modularity within neighbor neurons, the called *cell assemblies* (CAs), considering the high dimensional attribute and faculty of neurons [3]. It is a tendency of neurons to aggregate together with similar functional specializations and make organizations hierarchically. Though different classifications and names for neural clusters, we adopt the suggestion that *neuron - minicolumn - (hypercolumn) - macrocolumn - cortex area - hemisphere*, where minicolumn is a candidate for “the repeating pattern of circuitry” or “the iterated modular unit” [4].

In this paper, we exhibit a mathematical framework, noted briefly and named the fibre bundle map (FBM) methods in ref. [1], and show how to represent the neural process generally at the cortical level. Briefly speaking, the FBM representation is a mapping of the feature

components (often the synaptic weights) to topological spaces, the called *bundles*. The pattern informations are reordered in locally transformed coordinates and a few of major components are extracted. Obviously there exist another mathematical framework to represent the neural process in reduced space. Kohonen set up the mathematical preliminaries, the called feature maps or *feature-based* representation, in vector space and led the successive models in artificial and physiologic neural networks [5]. Symbolic processing architectures also suggest the description of neural computations at the cognitive and rational bands. The feature vector space or the symbolic sets can belong to a kind of FBM representation. But the FBM method has an interest in the manifold structure of frequent inputs in feature space and its corresponding symmetry group. Indeed, the properties of neural progress are governed not by the detailed neural interaction rules but by the algebraic structure of dominant feature components. The FBM method represents the neural process by the general formulas in statistical physics and helps to comprehend collective neural phenomena intuitively through the knowledge in statistical mechanics and differential geometry.

As a class of abstract representations, the formulas in the FBM models have some different character with those in the feature-based (often the called “low-dimensional” feature vector) models. In the feature-based representation, the change in the feature vector at position \mathbf{r} , $\Phi(\mathbf{r})$ is described as the difference in the stimuli vector $\Phi(\mathbf{r}')$, such as $\Delta\Phi(\mathbf{r}) \propto (\Phi(\mathbf{r}') - \Phi(\mathbf{r}))$ with the energy of the form $|\Phi(\mathbf{r}) - \Phi(\mathbf{r}')|^2$ (or its higher powers). Whereas, in the FBM representation, the interactions between neurons are notated by inner products rather than their distance, and it is generally assumed that the energy of neural process can be expanded in a power series, i.e.,

$$E[\psi] = E^{(0)} - \sum_i B(\mathbf{r}_i)\psi(\mathbf{r}_i)$$

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$$\begin{aligned}
& - \frac{1}{2!} \sum_{i,j} D(\mathbf{r}_i, \mathbf{r}_j) \psi(\mathbf{r}_i) \psi(\mathbf{r}_j) \\
& - \frac{1}{3!} \sum_{i,j,k} F(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \psi(\mathbf{r}_i) \psi(\mathbf{r}_j) \psi(\mathbf{r}_k) + \dots
\end{aligned} \tag{1}$$

where the field variables $\psi(\mathbf{r})$ denote the feature state of neurons at cortical location \mathbf{r} . We will show how this formula is derived and the interaction functions are determined in the simple (or the called “high-dimensional” feature vector representation) and the complex cell models. In a continuum limit, the energy can be approximated to

$$E[\psi] = \int d\mathbf{r} \left\{ \frac{v}{2} |(\nabla - i\mathbf{A})\psi|^2 + \frac{m^2}{2} |\psi|^2 + \frac{g}{4!} |\psi|^4 \right\}, \tag{2}$$

where the odd power terms are expected to be vanished generally. This is just the Ginzburg-Landau energy with gauge invariance and explains the statistical properties of the emergent cortical maps in experiments and simulations. The energy in a continuum approximation often can be derived using only minimal mathematical constraint such as the symmetry. The energy form in Eq.(1) and Eq.(2) proposes the possibility of the analogy between the physical and the neural systems, and the characteristics of developed visual maps are systematically apprehended through the statistical properties of vortices in magnetism [1, 2].

We apply the FBM representation method to the development models in visual cortex. The cortical map formation in orientation and ocular dominance columns is one of the most studied problems in brain. A considerable amount of different models is proposed, and some of which are compared with the experimental findings and in competition [6, 7]. Miller *et al.* formulated *correlation-based* models describing how ocular dominance and orientation columns develop in simple cell models [8, 9, 10]. Obermayer *et al.* presented a statistical-mechanical analysis of pattern formation and compared predictions quantitatively with experimental data using the Kohonen’s *self-organizing feature map* (SOFM) approaches. Wolf *et al.* obtained again the conditions for the emergence of a columnar pattern in the SOFM algorithm [11]. The studies of the *elastic-net* model also show the bifurcation and emergence of a columnar pattern [12, 13, 14]. Scherf *et al.* investigated the pattern formation in ocular dominance columns with more detailed model, which covers the results of the SOFM algorithm and the elastic-net model [15]. Wolf and Geisel predicted the influence of the interactions between ocular dominance and orientation columns on the pinwheel stability without model dependency and demonstrated it in the simulations of the elastic-net model [16]. The lateral (or neighbor) interaction models are also successful scheme based on physiology [1, 17, 18, 19].

In the proposed visual map formation models, the Hamiltonian models with spin variables belong to the class of FBM representation models [1, 19, 20]. Other development models written in the high- or low-dimensional

feature vector representation can be revised again in the FBM representation. The formulas in FBM models represent essential ingredients of neural interactions without paying much attention to particular neural control mechanism. Moreover, the modification of the iterative procedure of a model into the formula in Eq.(1) or Eq.(2) becomes the statistical analysis of the model itself. The quadratic interaction function $D(\mathbf{r}_i, \mathbf{r}_j)$ is consequence in the visual map formation as other physical systems. The interaction functions in neural process mean more than the intracortical connections or recurrences in the called lateral activity control. In the competitive Hebbian models, such as the elastic-net model and the SOFM algorithm, the interaction functions comprise the feedforward competition or normalization process. However, in the FBM representation the functional matrix $D(\mathbf{r}_i, \mathbf{r}_j)$ of the visual map formation models have common shape, the called Mexican hat type, that is, positive in short-range and negative in long-range, in spite of different development mechanisms. The bifurcation to an inhomogeneous state and the emergence of a columnar pattern is possible when there are strong negative interactions in $D(\mathbf{r}_i, \mathbf{r}_j)$. The development of a columnar pattern is also concerned with non-vanishing vector \mathbf{A} , the called *vector potential* in physics, in Eq.(2). The FBM representation method will show how the development models with different mechanisms lead to the successful formation of visual maps and share the statistical properties of vortices in the spin Hamiltonian models.

II. REPRESENTATION OF NEURAL STATE WITH COLUMNAR MODULARITY

The structures and connections in cerebral cortex are more complex and modular than those in artificial neural networks. Neurons tend to be vertically arrayed in the cortex, forming cylinders known as cortical columns. Traditionally, six vertical layers have been distinguished and classified into three different functional types. The layer IV neurons (*IN* box), first get the long-range input currents, and send them up vertically to layer II and III (*INTERNAL* box) that are the called true association cortex. Output signals are sent down to the layer V and VI (*OUT* box), and sent further to the thalamus or other deep and distant neural structures. Lateral connections also occur in the superficial (layer II and III) pyramidal neurons. In columnar (or horizontal) clustering, there are minicolumns, which are consisted of about 100 neurons and 30 μm in diameter in monkeys, and macrocolumns, which are 0.4~1.0 mm and contain at most a few hundred minicolumns. On the wider discrimination, there are 52 cortex areas in each human hemisphere; a Brodmann area averages 21 cm^2 and 250 million neurons grouped into several million minicolumns [4].

The columnar modules can be regarded as a kind of multi-layered neural networks and would have complex functional attributes. Most neurons in brain have the

attribute of *selective response* to a received activity, and the preferred signals become an useful representation of the functional attributes in a small neural region. A traditional representation of neural state is the vector notation \mathbf{v} , where its components correspond to the activity of each neuron in receptor layer. If a columnar module (or complex cell) at position \mathbf{r} respond selectively to a particular input vector \mathbf{v} and make an output vector \mathbf{y} , its functional attribute can be represented compactly as,

$$w(\mathbf{r}) = \mathbf{y} \circ F \circ \mathbf{v}^\top, \quad (3)$$

where F is the nonlinear response or activation function of complex cell. If the activation function is linear or ignored, this leads to a simple pattern associator, the called *linear associator*. The experiments of the response properties to external stimuli through electrode penetration can be understood as the measurement of the product between the associator $w(\mathbf{r})$ and the input signal \mathbf{v}' :

$$|w(\mathbf{r}) \circ \mathbf{v}'| = |\mathbf{y}| F(\mathbf{v}^\top \mathbf{v}'), \quad (4)$$

where the activity of the output $|\mathbf{y}|$ corresponds to the measurement of the number of action potentials or the frequency of spikes. In the physiological experiments with the complex cells in primary visual cortex [21] or the object perceptions in inferotemporal (IT) cortex [22], the response property of columnar modules used to be the combination of different patterns and then the functional form in Eq.(3) would be expanded into the summation of associators. When the output \mathbf{y} is common with the most favorite input \mathbf{v} such as Hopfield networks [23] or the most favorite input is only concerned, a vector notation can play the role of representation of functional attributes in columnar modules.

Fig.1 depicts a neural network with columnar modules. A matrix \mathbf{W} denotes the feedforward synaptic weights cross cortex areas, such as the connections between LGN and primary visual cortex, and the input vector to a columnar module is given by $\mathbf{v}_i = \mathbf{W}_i \mathbf{u}$ (or $v_i = \sum_\alpha W_{i\alpha} u_\alpha$). In a complex cell model, it is expected that the synaptic connections within a columnar module $w(\mathbf{r})$ achieve the functional attributes of neuron, whereas in a simple cell model, the connections with the external cells \mathbf{W} are considered to vest the functional attributes. For example, the ocular dominance in primary visual cortex is determined whether a neuron in V1 is more connected to the left or right eye (or LGN) cells. We call this the *extrinsic* information coding type, which is realized by the connectivity of far neurons cross cortex areas, whereas the *intrinsic* type is realized by the synaptic plasticity between close neurons within a columnar module. The neural attribute of two coding types are represented by common formula in FBM models, but there exist some different ground when building actual models. The feedforward competition behavior should be related to the intrinsic coding type. Moreover, the extrinsic encoding type causes a problem in modeling huge networks because too massive connections are required when the

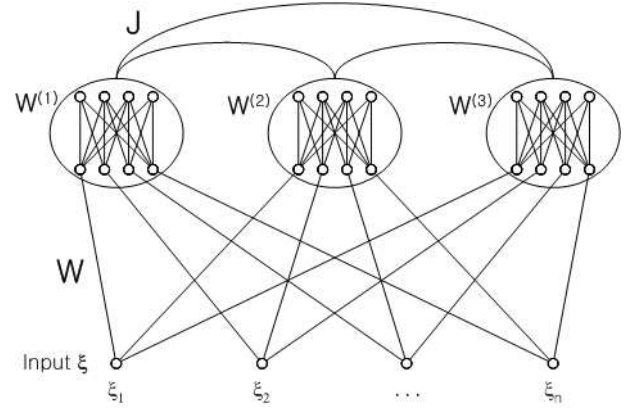


FIG. 1: A neural network model with columnar modules with function $w^{(i)}$. Input signal to a columnar module \mathbf{v}_i is driven by feedforward synapses with weights \mathbf{W} , that is, $\mathbf{v}_i = \mathbf{W}_i \mathbf{u}$, and its output \mathbf{y}_i is interconnected to neighbors by intracortical connections \mathbf{J} .

meaning of activity is characterized only from where the current come. We expect that the intrinsic type, encoding information in spatial or temporal correlations within a signal band, is essential in huge networks and would be a prominent strategy in the real brain.

III. FIBRE BUNDLE MAP REPRESENTATION

The FBM representation method bases on a mathematical framework - the called *fibre bundle* in manifold theory [24, 25]. For a trivial fibre bundle, a total (or bundle) space E , which will depicts the neural attributes at a cortical area, is composed of a base space B and a fibre F , tat is, $E = B \times F$. In our interests, cortical locations are the elements in base space, where feature (often pattern, code or model) space becomes a fibre. A structure (or symmetry) group G is a homeomorphism of fibre F , and the same with the fibre F in a *principal fibre bundle*. The principal fibre bundles admit *connexions* (or vector potentials in physics), and it is for this reason that they are of basic importance in gauge theories in physics. The features of cortical cells or small cortical regions at each cortical location \mathbf{r} are represented by a set of field variables $\psi_\alpha(\mathbf{r})$ and

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp(-i\phi_a(\mathbf{r})\tau^a) = \psi_a(\mathbf{r})\tau^a, \quad (5)$$

where $\phi_a(\mathbf{r})$ is an arbitrary internal (feature) phase and τ^a is the basis of a continuous (or Lie) group G . The bases can be taken as the amount of receptor cells, but are usually reduced according to the statistical structure of inputs. The frequent inputs usually occupy small regions in the total feature space and the major variance of feature components occurs within a embedded submanifold with high stimuli density (Fig. 2) The reduction of feature space is related to the extraction of features from

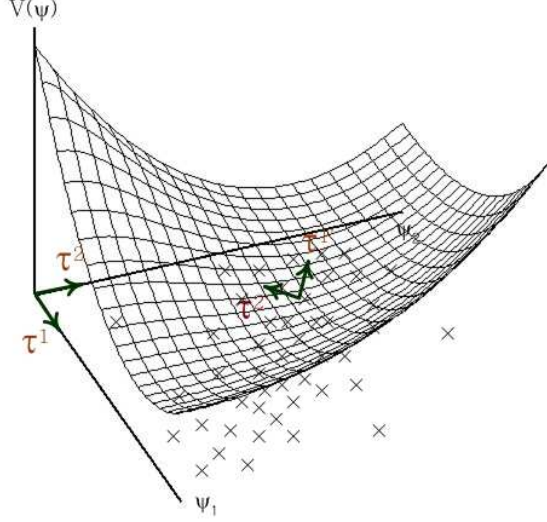


FIG. 2: Probabilistic external stimuli and a potential function with external source. The transformed basis $\tau^{1'}$ and $\tau^{2'}$ are the principal directions of external stimuli density at a point.

inputs in learning rules as well. Symmetry breaking between transformed feature components is expected in the neural progress of experience and learning, and cortical dynamics can be described with a few of field components in a reduced feature space.

The differential geometric concepts in the FBM representation furnish an intuitive explanation for emergent cortical maps. The self-organization of feature maps achieved by locally gathering similar interests means there is smooth variance of features with neighbor neurons at each location. In other words, the properties of “organized” and “optimized” feature maps is related with those of “continuous” and “flat” variables in manifold. If there is no difference of features with neighbors at small region near position \mathbf{r} , they can be denoted by $\nabla\psi(\mathbf{r}) = 0$ (or $\nabla\phi(\mathbf{r}) = 0$). If there exists small tilting of phase angle at position \mathbf{r} and an arbitrary vector $\mathbf{A}(\mathbf{r})$ denote the difference between phase angles, the called *covariant derivative* is given by $(\nabla - i\mathbf{A}(\mathbf{r}))\psi(\mathbf{r}) = 0$ (or $\nabla\phi(\mathbf{r}) - \mathbf{A}(\mathbf{r}) = 0$). If the covariant derivative vanishes (said to be flat or parallel by translated in manifold theory) for all \mathbf{r} , the distribution of the field variables $\psi(\mathbf{r})$ would be a minimum solution of the integral

$$S = \int d\mathbf{r} |(\nabla - i\mathbf{A})\psi|^2, \quad (6)$$

for the connexion \mathbf{A} . A non-vanishing connexion \mathbf{A} occurs when there are strong competitive behavior or inhibitory lateral interactions between neurons, and is related to the emergence of a periodic pattern in cortical maps, such as the band patterns in ocular dominance columns and the linear zones in orientation preference

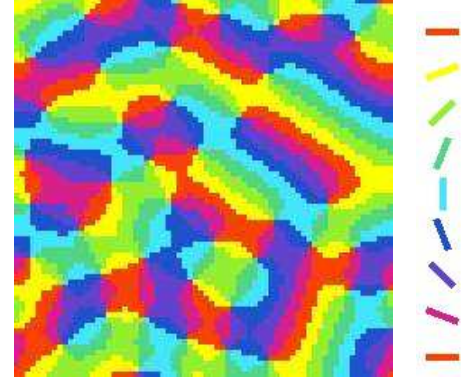


FIG. 3: The simulation result of orientation map formation. The orientation maps have $U(1)$ (or $O(2)$) symmetry and the major characteristics of the developed map can be predicted using only the symmetry properties.

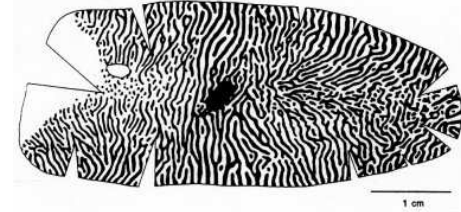


FIG. 4: The complete pattern of ocular dominance stripes in the striate cortex of a macaque monkey. There is a strong tendency for the stripes to meet the margin of striate cortex at steep or right angles. (Reprinted by permission from S.LeVay, Copyright ©1985 by the Society for Neuroscience [26].)

columns, with the wavelength $\Lambda = 2\pi/|\mathbf{A}|$. Fig.4 shows the complete pattern of ocular dominance stripes of a macaque monkey. The orthogonality between the contour lines of feature map and the boundary of cortical area is a property of minimal solutions in Eq.(6). From the condition $\delta S/\delta\phi \sim 0$ or $\nabla^2\phi \sim 0$ for $\psi = e^{2i\phi}$ with the preferred angle ϕ , the normal component of $\nabla\phi$ vanishes at the area boundary since the integral along a narrow rectangular loop over the area boundary $\oint_C \nabla\phi \cdot d\hat{n}$ vanishes due to the divergence theorem. Such perpendicularity with the area boundary is also manifested in other static field solutions, such as the magnetic field.

The symmetry property also helps to predict the energy function of the cortical map formation. For example, the features of orientation preference columns in the visual cortex have $U(1)$ (or $O(2)$) symmetry. However we perform a rotation in all the preferred angles through same angle ($\phi \rightarrow \phi + \chi$ - called ‘global’ gauge transform), the energy of orientation map formation should remain invariant. The rotation angle χ can have a dependency on position \mathbf{r} , the called ‘local’ gauge transform, and the energy in a continuum approximation may take the form in Eq.(2) or Eq.(6) with $\mathbf{A} = \nabla\chi(\mathbf{r})$.

IV. DESCRIPTION OF DETAILED NEURAL INTERACTIONS

The description of neural dynamic at a high level also should be based both on neuroscience and informatics. One important principle for neural plasticity is the Hebbian rule : two simultaneously active neurons on either side of a connection increases the weight of that connection is increased [3]. The simple Hebbian plasticity rule in a single neuron consists of inputs \mathbf{u} and weights \mathbf{W} takes the form

$$\Delta \mathbf{W}(t) \propto y(t)\mathbf{u}(t) \quad (7)$$

for the output $y = f(\mathbf{W}\mathbf{u})$ with the activation function f of simple cell. In intracortical connected networks, the input becomes the summation of the current from input and neighbor cells. The output of neuron at i -th site becomes

$$y_i = f(v_i + \sum_j J_{ij}y_j) \quad (8)$$

for $v_i = W_{i\alpha}u_\alpha$ and the recurrent weight matrix \mathbf{J} . In a energy model, synaptic plasticity rule is regarded as the negative gradient of an energy (often objective, error or cost function) defined as a function of \mathbf{W} :

$$\Delta \mathbf{W} \propto -\frac{\partial E[\mathbf{W}]}{\partial \mathbf{W}}. \quad (9)$$

Because of the nonlinearity of the activation function and the recursive form in Eq.(8), the energy used to be approximated depending on models. For example, with the assumption of $y_i = f(v_i + \sum_j J_{ij}v_j)$ and a series expression of activation function $f(v) = \sum_\ell a_{\ell+1}v^\ell$, the energy is obtained by

$$E[\mathbf{W}] = -\sum_\ell \frac{a_\ell}{\ell} D_{i_1 \dots i_\ell}^{(\ell)} Q_{\alpha_1 \dots \alpha_\ell}^{(\ell)} W_{i_1 \alpha_1} \dots W_{i_\ell \alpha_\ell}, \quad (10)$$

where

$$\begin{aligned} D_{i_1 \dots i_\ell}^{(\ell)} &= (\delta_{i_1 i_2} + J_{i_1 i_2}) \dots (\delta_{i_{\ell-1} i_\ell} + J_{i_{\ell-1} i_\ell}) \\ &= D_{i_1 i_2}^{(2)} \dots D_{i_{\ell-1} i_\ell}^{(2)} \end{aligned} \quad (11)$$

is the functional tensor of rank ℓ and

$$Q_{\alpha_1 \dots \alpha_\ell}^{(\ell)} = \langle u_{\alpha_1} \dots u_{\alpha_\ell} \rangle_{\mathcal{D}} \quad (12)$$

is the input correlation tensor of rank ℓ . $\langle \cdot \rangle_{\mathcal{D}}$ denotes the average over input data set \mathcal{D} . This energy based on the basic Hebbian rule is adjusted again depending on the characteristic of synaptic plasticity rules [27]. For example, the covariance plasticity rule replaces the input correlation function $\mathbf{Q}^{(\ell)}$ with rank ℓ as the input covariance function

$$C_{\alpha_1 \dots \alpha_\ell}^{(\ell)} = \langle (u_{\alpha_1} - \langle u_{\alpha_1} \rangle_{\mathcal{D}}) \dots (u_{\alpha_\ell} - \langle u_{\alpha_\ell} \rangle_{\mathcal{D}}) \rangle_{\mathcal{D}}. \quad (13)$$

In the FBM representation of simple cell model, feedforward synaptic weights $\mathbf{W}_{i\alpha}$ is replaced as field variables $\psi_\alpha(\mathbf{r}_i)$, then the energy in Eq.(10) satisfies the form of energy in Eq.(1). For efficient description of dynamics, the energy is decomposed into the functions of transformed field variables. Because of the anisotropy in input correlation \mathbf{Q} (often in neighbor activity \mathbf{D}), the symmetry between components is broken and the effective dynamics can be described with a few of dominant components. The consequence of the anisotropy in neighbor activity between feature components is explored in the case of the anisotropy between orientation and ocular dominance columns [2].

In a complex cell model, the features of neurons relate to the synapses within a columnar module. The columnar module is a kind of adaptive neural network systems and the modulation of its functional attributes involves intricate changes in synaptic weights. An effective assumption is that the output of a columnar module is one of the proper states of the functional and will change following afferent signals. For the currents from input and neighbor cells and a linear activation function, the change in the proper state or the output of a columnar module is then

$$\Delta \mathbf{y}_i \propto \mathbf{v}_i + \sum_j J_{ij} \mathbf{y}_j \quad (14)$$

for the input \mathbf{v}_i to the columnar module at position \mathbf{r}_i and the energy averaged over inputs is obtained by

$$E[\mathbf{y}] = -\sum_i \langle \mathbf{v}_i \rangle_{\mathcal{D}} \mathbf{y}_i - \frac{1}{2} \sum_{i,j} J_{ij} \mathbf{y}_i \mathbf{y}_j. \quad (15)$$

In the FBM representation, the output with multivariable is replaced by field variables :

$$E[\psi] = -\sum_i B_i \psi_i - \frac{1}{2} \sum_{i,j} J_{ij} \psi_i \psi_j \quad (16)$$

or

$$\begin{aligned} E[\psi] &= -\sum_i B(\mathbf{r}_i) \psi^\dagger(\mathbf{r}_i) \\ &\quad - \frac{1}{4} \sum_{i,j} J(\mathbf{r}_i, \mathbf{r}_j) \{ \psi(\mathbf{r}_i)^\dagger \psi(\mathbf{r}_j) + \psi(\mathbf{r}_i) \psi(\mathbf{r}_j)^\dagger \}, \end{aligned} \quad (17)$$

where a functional vector $B(\mathbf{r}_i) = \langle \mathbf{v}_i \rangle_{\mathcal{D}}$ is the linear average over inputs. If we assume ψ^\dagger and ψ are creation and annihilation operators, the term $\psi(\mathbf{r}_i) J(\mathbf{r}_i, \mathbf{r}_j) \psi^\dagger(\mathbf{r}_j)$ can be regarded as the description of phenomena that a created activity at position \mathbf{r}_j is translated with kernel J and annihilated at position \mathbf{r}_i .

A series of physiological experiments showed that the synaptic plasticity comes from a redistribution of the available synaptic efficacy, not an increase in the efficacy [27, 28]. In other words, the neural plasticity at the network level can be understood as the pursuit of increment in the probability of reactivity with bounded total

synaptic strength for environmental experience. With the expectation of a automatic normalization of synaptic weights, the norm of field variables $|\psi|$ used to be constrained to be constant. In this sense, the neural dynamics with functional modularity may be described by the slight shift in the internal phase per activity following afferent signals. Sometimes the normalization constraint is not imposed and involved in the plasticity rule with subtractive normalization [29]. For the energy function of the form

$$E[\psi] = a\psi^2 - b\psi^4, \quad (18)$$

the stability of synaptic weight can be achieved due to the relaxation of $|\psi|^2$ to its equilibrium value.

Another important mechanism expected in neural computation is the enhancement of neural activity depending on correspondence to input. A possible enhancement modulation is the restriction on the sum over the activity by subtractive normalization. With a simple nonlinear form $x + \eta x^2$, the external source term with enhanced afferent signals becomes that

$$\begin{aligned} \langle \mathbf{v}'_i \rangle_{\mathcal{D}} \psi_i &= \left\langle \mathbf{v} \frac{\rho_i(1 + \eta \mathbf{v} \psi_i)}{(1/\rho) \sum_j \rho_j(1 + \eta \mathbf{v} \psi_j)} \right\rangle_{\mathcal{D}} \psi_i \\ &\simeq \langle \mathbf{v}_i \rangle_{\mathcal{D}} \psi_i + \frac{1}{2} \sum_j S_{ij} \psi_i \psi_j \end{aligned} \quad (19)$$

for $\mathbf{v}_i = \rho_i \mathbf{v}$ and $\rho = \sum_i \rho_i$ with the stimuli strength ρ_i at position \mathbf{r}_i . The scattering function with a input data set \mathcal{D} is defined as

$$S_{ij} = 2\eta \langle v_i v_j \rangle_{\mathcal{D}} (\delta_{ij} - 1) \quad (20)$$

for the enhancement (or competition) parameter η . In the FBM representation, the scattering function describes the feedforward competition process in the competitive Hamiltonian models, such as the elastic-net model and the SOFM algorithm. In fact, for an intrinsic coding type, network cannot tell which neurons match mostly with input signal a priori and the winner has to be determined after lateral inhibitory activity. The competitive Hebbian models require a normalization control of response or a priori decision of winner and depict the feature vectors in visual cortex through the connectivity between visual cortex and retinas (or LGNs) [15]. The lateral activity function $J(\mathbf{r}_i, \mathbf{r}_j)$, the connectivity between neurons (or columnar modules) at position \mathbf{r}_i and \mathbf{r}_j within a cortex area, has two types according to the control mechanisms [30]. In the case of the lateral feedback control (which Kohonen called the activity-to-activity kernel), the lateral activity function \mathbf{J} is regarded to be excitatory for short distance and inhibitory for long distance with the so-called Mexican hat type (Fig.5a). Whereas in the case of the lateral control of plasticity (or the activity-to-plasticity kernel), the lateral interaction is nonnegative and may take the Gaussian form (Fig.5b). The competitive Hebbian models take the lateral control of plasticity, that means there is no negative value

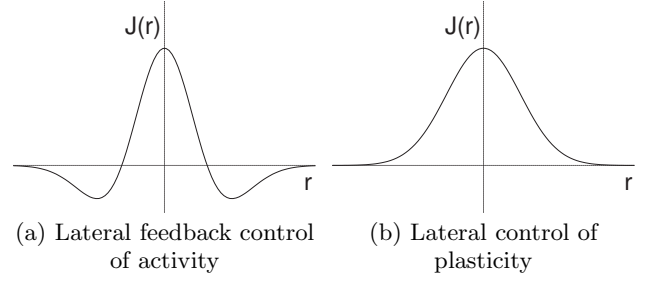


FIG. 5: The two types of neighbor interaction functions and control mechanisms. (a) The lateral interaction models adopt lateral activity control and the activation kernel, usually so-called “Mexican hat” function (positive feedback for close distance and negative for longer distance). (b) The plasticity control with nonnegative kernel requires feedforward competition (or feedforward normalization of activity over networks). The elastic-net model assumes the nearest neighbor interactions (or elastic force), whereas the SOFM algorithm takes the neighbor function on Gaussian form with the hard competition (or winner-take-all activity).

in \mathbf{J} , and the scattering function \mathbf{S} from afferent signal enhancement has an effect of inhibitory activity.

Now we employ the concepts of thermodynamic into neural dynamics. In some classes of neural network models, such as Boltzmann machine, the input-output is assumed to be stochastic. Once a stochastic neural network has converged to an equilibrium state, the probability distribution characterizing ψ is expected to obey the Boltzmann distribution

$$P[\psi] = \frac{\exp(-E[\psi])}{Z} \quad (21)$$

for the partition function

$$Z = \sum_{\psi} \exp(-E[\psi]). \quad (22)$$

In neural processing architecture, the notion of entropy or free energy is put into practice ahead for the purpose of informatics. Compared to deterministic firing models, an expected advantage in stochastic neural network models is to escape from poor locally optimal configurations through probabilistic evolution. Moreover, there are several reasons that the stochastic behavior should be indispensable process in neural networks. In view of learning rules, it is natural that neural states are occupied with features corresponding to frequent inputs (the *coarse coding* principle). On the other hand, it is efficient for a neural network to avoid the occupation with a few of features, so that an object is coded by a small population that is active for an event (the *sparse coding* principle). It is usual that the cost function in unsupervised learning algorithm is similar to the Helmholtz free energy that

$$F = E - TS, \quad (23)$$

where the parameter T is considered just as a positive constant that determines the importance of the second term relative to the first. In learning rules, the energy term is illustrated by a measurement how well the code describes the input data or carry the informations :

$$\begin{aligned} E &= (1/N) \sum_i \sum_{\mathbf{v} \in \mathcal{D}} P(\psi_i | \mathbf{v}) P(\mathbf{v} | \mathcal{D}) \\ &= (1/N) \sum_i P(\psi_i | \mathcal{D}) \\ &= \sum_{\psi} P(\psi) P(\psi | \mathcal{D}), \end{aligned} \quad (24)$$

where a distribution $P(\psi | \mathcal{D})$ is the average over the probability that input $\mathbf{v} \in \mathcal{D}$ generates output ψ . In Hebbian development models, this energy term can be considered as an external source term, that is the average over the product between feature state and external signals, $-B\psi$ in a complex cell model (or $-\psi \mathbf{Q}^{(2)} \psi$ in a simple cell model) as well. In learning rules, the entropy term assesses the sparseness of the code by assigning a cost depending on how the activity is distributed. According to Shannon's coding theorem, the amount of information is defined by

$$S = -K \sum_{\psi} P(\psi) \ln P(\psi) \quad (25)$$

where K is a positive constant and $-\ln P(\psi)$ is the cost of the code, the number of bits required to communicate the code. The connections between information theory and statistical mechanics are rigorously investigated [31, 32, 33]. However, there is some hardship to apply the statistical mechanism to the phenomena in the real brain. Since neural process, comprehends the dynamics at various spatial and temporal levels, is essentially dynamical and non equilibrium phenomenon. For example, the relaxation process in cortical map formation is very slow and the observed maps often do not satisfy the equilibrium criteria. The map formation in visual cortex occurs concentrately for several weeks or months after birth, during a so-called critical period. In observed orientation preference maps, the non-uniforming directions of gradient ($\nabla \phi_{\parallel} \neq \text{const}$, however $|\nabla \phi_{\parallel}| \simeq \text{const}$ for the longitudinal component ϕ_{\parallel}) and non-vanishing singular points (or pinwheels) indicate that the system may be frozen during the relaxation process [1].

V. APPLICATION TO VISUAL MAP FORMATION MODELS

According to the studies of the statistical structure of natural images, the response properties of visual neurons, the spatially localized and oriented, are considered to be due to the efficient coding of natural images [34]. Oriented bar or grid patterns are the most probable activity and the feature with $O(2)$ (or $U(1)$)

symmetry components is a meaningful representation in orientation columns. With ocular dominance columns, the total feature can be expanded to $O(3)$ symmetry components with the restriction of synaptic normalization within each column. Therefore, the conventional spin vector (S^x, S^y, S^z) can serve as a useful representation of the feature states with the preferred orientation $\phi = (1/2) \tan^{-1}(S_x/S_y)$. The proposed models of visual map formation are based on so various mechanisms. We rewrite four widely used visual map development models in term of FBM representation classified by their effective interaction terms that

- (A) Lateral interaction models : $\mathbf{D} = \mathbf{J}$
- (B) Recursive interaction models : $\mathbf{D} = (\mathbf{I} - \mathbf{J})^{-1}$
- (C) Elastic-net model : $\mathbf{D} = \mathbf{J} + \mathbf{S}$
- (D) SOFM algorithm : $\mathbf{D} = \mathbf{S}\mathbf{J}$.

The lateral activity function $J(\mathbf{r}_i, \mathbf{r}_j)$ in the elastic-net model and the SOFM algorithm is taken to be all non-negative, the two-point interaction function $D(\mathbf{r}_i, \mathbf{r}_j) \simeq D(|\mathbf{r}_i - \mathbf{r}_j|)$ takes the Mexican hat type for all cases owing to the scattering function $S(\mathbf{r}_i, \mathbf{r}_j)$. Periodic patterns, such as linear zones in orientation preference columns or parallel bands in ocular dominance columns, can develop when there are abundant negative values in $D(\mathbf{r}_i, \mathbf{r}_j)$ so that $\tilde{D}(q)$ in Fourier space has a non-vanishing minimum point q^* with the wavelength $\Lambda = 2\pi/q^*$.

A. Lateral Interaction Models

A simple cell model of the visual map development uses the high-dimensional feature vector coding for the strength of the connection from each cortical location to each retinal (or LGN) location. The synaptic plasticity depends on the average over the activities of competing inputs, which are left and right eyes for ocular dominance columns or ON-center and OFF-center cells for orientation preference columns. For a linear activation function or $f(v) = v$, the energy in Eq.10 becomes that

$$E[\mathbf{W}] = -\frac{1}{2} \sum_{i,j} \sum_{\alpha,\beta} (\delta_{ij} + J_{ij}) Q_{\alpha\beta}^{(2)} \mathbf{W}_{i\alpha} \mathbf{W}_{j\beta}. \quad (26)$$

This energy is decomposed with the (globally) transformed synaptic weights into the sum and difference :

$$\mathbf{W}_+ = \mathbf{W}_R + \mathbf{W}_L \quad \text{and} \quad \mathbf{W}_- = \mathbf{W}_R - \mathbf{W}_L \quad (27)$$

for ocular dominance columns or similarly $\mathbf{W}_{\pm} = \mathbf{W}_{ON} \pm \mathbf{W}_{OFF}$ for orientation columns. In a pixel-based representation for orientation columns, oriented patterns with low-frequency compose a dominant feature space and the energy is decomposed with the locally transformed weights as well. Therefore, the energy as the

function of field variables ψ in a transformed and reduced feature space is that

$$E[\psi] = -\frac{1}{2} \sum_{i,j} D(\mathbf{r}_i, \mathbf{r}_j) \psi(\mathbf{r}_i) \psi(\mathbf{r}_j) \quad (28)$$

for $\mathbf{D} = \mathbf{I} + \mathbf{J}$. The input correlation matrix $\mathbf{Q}^{(2)}$ is ignored, because the frequency of the dominant features in inputs is regarded to be the same or the two-point activity function \mathbf{D} comprises this. The term $-\frac{1}{2} \sum_i \psi(\mathbf{r}_i)^2$, related to the self-relaxation term, does not effect influence on the typical spacing of an emergent columnar pattern. In a complex cell model, the simplest is given by the summation of the neighbor interactions and the external stimuli terms as

$$E[\psi] = -\frac{1}{2} \sum_{i,j} D(\mathbf{r}_i, \mathbf{r}_j) \psi(\mathbf{r}_i) \psi(\mathbf{r}_j) - \sum_i B(\mathbf{r}_i) \psi(\mathbf{r}_i) \quad (29)$$

for $\mathbf{D} = \mathbf{J}$. The external stimuli $B(\mathbf{r}_i)$ is considered to be constant or vanishing. Therefore, the form of the lateral activity function \mathbf{J} determines the typical appearance of developed feature map for both cases. In lateral interaction models, \mathbf{J} is taken as the *activation kernel*, or Mexican hat function (positive feedback in the center, negative in the surroundings). For example, a well-known Mexican hat function, the called difference of Gaussians (DOG) filter, is described as

$$J(\mathbf{r}_i, \mathbf{r}_j) = \varepsilon \left(e^{-|\mathbf{r}_i - \mathbf{r}_j|^2 / 2\sigma_1^2} - k e^{-|\mathbf{r}_i - \mathbf{r}_j|^2 / 2\sigma_2^2} \right) \quad (30)$$

where k is the strength of inhibitory activity. Another example of Mexican hat function modified from a wavelet is given by

$$J(\mathbf{r}_i, \mathbf{r}_j) = \varepsilon \left(1 - k \frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{\sigma_l^2} \right) e^{-|\mathbf{r}_i - \mathbf{r}_j|^2 / 2\sigma_l^2} \quad (31)$$

for the lateral cooperation range σ_l . If the strength of inhibitory activity k is larger than threshold $k_c (= 1/4)$, $\tilde{D}(q)$ has a non-vanishing maximum point at $q^* = (1/\sigma) \sqrt{4 - 1/k}$ [1].

B. Recursive interaction models

For a linear activation function, the output in Eq.(8) becomes

$$y_i = v_i + \sum_j J_{ij} v_j + \sum_{j,k} J_{ij} J_{jk} v_k + \dots, \quad (32)$$

which is the summation of recursive recurrences. The energy as the function of synaptic weights is obtained that

$$E[\mathbf{W}] = -\frac{1}{2} \sum_{i,j} \sum_{\alpha,\beta} D_{ij} Q_{\alpha\beta}^{(2)} \mathbf{W}_{i\alpha} \mathbf{W}_{j\beta}, \quad (33)$$

where the two-point interaction function is

$$\mathbf{D} = \mathbf{I} + \mathbf{J} + \mathbf{J}^2 + \dots = (\mathbf{I} - \mathbf{J})^{-1} \quad (34)$$

and the real parts of the eigenvalues of \mathbf{J} are expected to be less than 1. Eq.(33) is a simple modified equation of Miller's correlation-based learning models [35]. In the original representation by Miller *et al.*, the input stimuli term is described by an arbor function, expressing the location and the overall size of the receptive fields [8, 10]. The two-point interaction function \mathbf{D} takes the Mexican hat type and the wavelength of visual pattern is determined by the peak of $\tilde{D}(q)$ in the analysis by Miller as well [36].

C. Elastic-Net Model

The elastic-net model is described by an iterative procedure with the update rule :

$$\Delta\Phi(\mathbf{r}_i) = \alpha \sum_{|\mathbf{r}_i - \mathbf{r}_j| = a} (\Phi(\mathbf{r}_i) - \Phi(\mathbf{r}_j)) + \beta(\mathbf{V} - \Phi(\mathbf{r}_i)) \frac{e^{-|\mathbf{V} - \Phi(\mathbf{r}_i)|^2 / 2\sigma_s^2}}{\sum_j e^{-|\mathbf{V} - \Phi(\mathbf{r}_j)|^2 / 2\sigma_s^2}}, \quad (35)$$

where a feature vector in the low-dimensional representation is

$$\Phi(\mathbf{r}) = (r_x, r_y, q \sin(2\phi(\mathbf{r})), q \cos(2\phi(\mathbf{r})), z(\mathbf{r})) \\ = (\mathbf{r}, \psi(\mathbf{r}))$$

for the retinal location $\mathbf{r} = (r_x, r_y)$, the preferred orientation $\phi(\mathbf{r})$, the degree of preference for that orientation and the ocular dominance z [6, 12]. At each iteration, a stimulus vector $\mathbf{V} = (\mathbf{r}_v, \mathbf{v})$ is chosen at random according to a given probability distribution. The first term in Eq.(35) denotes the elastic force or the excitatory interactions between the nearest-neighbors, and the second term implies the normalized stimuli distributed around an activity center. Functional Taylor expansion of the right hand side after dropping all nonlinear terms leads to

$$\Delta\psi(\mathbf{r}_i) = \alpha \sum_{|\mathbf{r}_i - \mathbf{r}_j| = a} \{ \psi(\mathbf{r}_i) - \psi(\mathbf{r}_j) \} - \beta \psi(\mathbf{r}_i) + \frac{\beta a^4}{4\pi^2 \sigma_s^6} \sum_j \langle \mathbf{v}_i \mathbf{v}_j \rangle_{\mathcal{D}} \{ \psi(\mathbf{r}_i) - \psi(\mathbf{r}_j) \} \quad (36)$$

where the stimulus at position \mathbf{r}_i ,

$$\mathbf{v}_i = \mathbf{v} e^{-|\mathbf{r}_i - \mathbf{r}_v|^2 / 2\sigma_s^2} \quad (37)$$

is distributed in a gaussian form with the activity center \mathbf{r}_v and the feedforward cooperation range σ_s . The correlation between the external stimuli at position \mathbf{r}_i and \mathbf{r}_j is obtained by

$$\langle \mathbf{v}_i \mathbf{v}_j \rangle_{\mathcal{D}} = \langle v^2 \rangle_{\mathcal{D}} \sum_{\mathbf{r}_v} e^{-|\mathbf{r}_i - \mathbf{r}_v|^2 / 2\sigma_s^2} e^{-|\mathbf{r}_j - \mathbf{r}_v|^2 / 2\sigma_s^2} \\ \simeq (\pi \sigma_s^2 / a^2) \langle v^2 \rangle_{\mathcal{D}} e^{-|\mathbf{r}_i - \mathbf{r}_j|^2 / 4\sigma_s^2}. \quad (38)$$

Therefore, the effective energy of the elastic-net model can be represented through the form in Eq.(29), where the two-point interaction function is given by

$$\mathbf{D} = -\beta\mathbf{I} + \mathbf{J} + \mathbf{S}. \quad (39)$$

The lateral activity function becomes $J(\mathbf{r}_i, \mathbf{r}_j) = \alpha\delta(|\mathbf{r}_i - \mathbf{r}_j| - a)$ or the Laplacian operator in a continuum limit. The scattering function coincides with the form in Eq.(20) for $\eta = \beta a^4/8\pi\sigma_s^6$ or is obtained by

$$S(\mathbf{r}_i, \mathbf{r}_j) \simeq \frac{\beta}{\sigma_s^2} \langle v^2 \rangle_{\mathcal{D}} \left(\delta_{ij} - \frac{a^2}{4\pi\sigma_s^2} e^{-|\mathbf{r}_i - \mathbf{r}_j|^2/4\sigma_s^2} \right). \quad (40)$$

This result means that the scattering function $S(\mathbf{r}_i, \mathbf{r}_j)$ can act as a kernel with inhibitory activity however the lateral activity function $J(\mathbf{r}_i, \mathbf{r}_j)$ is nonnegative. There are also interaction terms of higher power but the two-point interaction function $D(\mathbf{r}_i, \mathbf{r}_j)$ determines the major characteristics of developed feature maps. We transform it to Fourier space and obtain

$$\begin{aligned} \tilde{D}(\mathbf{q}) &= -\beta + \tilde{J}(\mathbf{q}) + \tilde{S}(\mathbf{q}) \\ &\simeq -\beta - \alpha q^2 + \frac{\beta}{\sigma_s^2} \langle v^2 \rangle_{\mathcal{D}} \left(1 - e^{-q^2 \sigma_s^2} \right). \end{aligned} \quad (41)$$

It has a maximum at

$$q^* = \frac{1}{\sigma_s} \sqrt{\ln \left(\frac{\beta}{\alpha} \langle v^2 \rangle_{\mathcal{D}} \right)}, \quad (42)$$

which corresponds to the analytic results from different approaches [13, 15].

D. Self-Organizing Feature Map Algorithm

In Eq.(29), the interaction term $\psi J \psi$ denotes the exchange of spontaneous spikes, created without external activity. Spontaneous firings can occur in coupled nonlinear oscillators with small dynamic fluctuations, which have been observed in some experiments [37, 38, 39, 40, 41, 42]. However, several experiments suggested that the organization of feature maps is possible after the exposure to the external activity. In this case, the probability of spontaneous firing are small ($J \ll v$), so that the most intracellular interactions would be achieved by indirect currents of external activities. With the provoked interactions by external activities, we can take the effective energy as

$$E[\psi] = - \left(\sum B\psi + \frac{1}{2} \sum \psi S \psi \right) \left(\frac{1}{2} \sum \psi J \psi \right). \quad (43)$$

If $B(\mathbf{r})\psi(\mathbf{r})$ is constant for all position \mathbf{r} , the first term with $\psi J \psi$ supports the lateral interaction models again. In the Kohonen's SOFM algorithm, the lateral currents induced by feedforward normalized stimuli are focused and the effective interaction term is given by

$$D(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{2} \sum_{\mathbf{r}} S(\mathbf{r}_i, \mathbf{r}) J(\mathbf{r}, \mathbf{r}_j). \quad (44)$$

Moreover, the SOFM algorithm requires the hard competition, the called "winner take all" (WTA) case. As σ_s approaches zero (or large η), the activity is localized only around the winning neuron and the scattering function in Fourier space becomes $\tilde{S}(\mathbf{q}) \simeq \beta \langle v^2 \rangle_{\mathcal{D}} q^2$, the Laplacian operator. The lateral activity function in the SOFM approaches takes on the Gaussian form $J(\mathbf{r}_i, \mathbf{r}_j) = e^{-|\mathbf{r}_i - \mathbf{r}_j|^2/2\sigma_l^2}$ for the lateral cooperation range σ_l (lateral plasticity control). Therefore we obtain the two-point interaction function

$$\tilde{D}(\mathbf{q}) = \frac{1}{2} \tilde{S}(\mathbf{q}) \tilde{J}(\mathbf{q}) = \pi \sigma_l^2 \beta \langle v^2 \rangle_{\mathcal{D}} q^2 e^{-q^2 \sigma_l^2/2} \quad (45)$$

in Fourier space or

$$D(\mathbf{r}_i, \mathbf{r}_j) = \beta \langle v^2 \rangle_{\mathcal{D}} \left(1 - \frac{|\mathbf{r}_i - \mathbf{r}_j|^2}{2\sigma_l^2} \right) e^{-|\mathbf{r}_i - \mathbf{r}_j|^2/2\sigma_l^2} \quad (46)$$

in real space. This is the Mexican hat function in Eq.(31) with $k = 0.5$. Eq.(45) has a minimum at

$$q^* = \sqrt{2}/\sigma_l, \quad (47)$$

which agrees with previous analytic results [11, 15, 43] and always positive if $\sigma_l > 0$. The Kohonen's SOFM algorithm reads to robust learning rules because it always succeeds in achieving an array of different feature detectors or a columnar pattern.

VI. DISCUSSION

The physical models of neural network based on neuroscience attempt to interpret both physiologic phenomena and computational architectures. In order to study the functional of the real brain, we need more adaptable theories than the basic neural architecture with connectionism. In this paper, we show that the neural process at the cortical level can be described by using the conventional expressions in statistical physics. As we showed in visual map formations [1, 2], the collective neural dynamics can be much alike well-known phenomena in the physical systems.

In the assumption of neural network composed of columnar modules, we classify the synaptic connection types and anticipate different functional characters in computational processing. (1) In the connectivity between close neurons within a columnar module, the functional attributes of neurons and the associative memory is realized. (2) By the connectivity between columnar modules within a cortex area, noted by the lateral activity function or recurrent weight matrix \mathbf{J} , the networks control laterally the output activity between neighbors. (3) Via the connectivity between far apart neurons cross cortex areas, neurons get driven-activity from external environment or other functional cortex areas. The columnar modules become elements (or nodes) again with high dimensional attributes in networks of neural networks. If

the recurrent weights matrix \mathbf{J} is specified depending on the positions, the connectivity between columnar modules also work in information coding. The connectivity between columnar modules within or beyond cortex areas would be strengthened also if there are much communications between them according to the Hebbian rule, and there are some models holding the updating rule in the recurrent weights matrix \mathbf{J} , such as the Goodall rule. [44]. We regard that the enhancement of connectivity between columnar modules proceed to the efficient communications between neurons rather than information coding. The consideration of minicolumn as a columnar module and the processing element in network is optional. The formation of structure in minicolumn is also due to the functional grouping between neurons with similar interests, and expected to be certified with more fundamental process at the cellular or molecular level.

Extraction of the significant features in the input data is the purpose of an unsupervised learning rule and also expected to be a principle character of artificial and physiologic neural networks. The FBM representation method suggests how neurons find features from afferent signals and build knowledge at the cortical level. An abstract representation of features in the FBM representation and a symmetry breaking between feature components in progress is related to the learning process in the neural network. For example, difference looks of an object form a submanifold in pattern space and the patterns of the object can be abstracted and decomposed in

the transformed and reduced feature space.

In view of dynamics, the essential factors in neural process are (1) statistical structure of inputs, (2) attractive or repulsive interactions between neighbor neurons, and (3) stochastic behavior of neurons. In this paper, we did not fully apply thermodynamic mechanics into neural process. There are some models which contain thermodynamic approach. The basic ingredients of Tanaka's Potts spin models are those of the lateral interaction models but he took a probabilistic evolution rather than a energy gradient flow [20, 45, 46, 47, 48]. Piepenbrock presented a model which uses the effect of stochastic behavior in neural network as a competition process [49]. However there is no lateral inhibitory activity or feedforward competition, thermodynamic effect can make a network to have a columnar structure with a thermal excitation at low temperature. We expect the stochastic behavior of neurons can be the connection between the physical neural dynamic models and the neural network models originated from learning theory and an essential factor in comprehension of systematic ordering-disordering or bifurcation problems in the real brain. Moreover, we expect that the theoretic experience in physics can offer more intuitive appreciation of the physiologic phenomena at higher level and sophisticated mechanisms in computational architecture.

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